Actions and invariant schemata in linear generalising problems

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In this paper we outline some results obtained from an ongoing research on the students' process of generalization. A written test and a task based interview was administered to eleven secondary students (aged 15-16 years). The teorethical model for generalization developed by Dörfler (1991) has probed to be usefull in analyzing the students' processes. We report briefly actions performed and invariant schemata established by students related to two different setting: numerical and spatial.

In its usual presentation, a linear generalising problem is a word problem that includes the first three terms of a numerical sequence and some pictures to illustrate the situation described. From a mathematical point of view we have an affin function $f(n)=dn+b$, $(b\neq 0)$, and there is an essential relationship namely $f(n+1)-f(n)=d$, to say that the difference between consecutive terms in the sequence is a constant.

Current research on student perception and generalization of numerical pattern has identified and classified the strategies used to solve linear generalising problems (Stacey, 1989; Orton & Orton, 1994 and 1996; García-Cruz & Martinón, 1996a). The SOLO taxonomy was used by Redden (1994) to classify the students' responses to a written test and to set a developing model for generalization.

However, little attention has been paid to the process through which one the students construct and develop a generalization in these types of problems. The role played by the drawing in the generalization process has been partially sketched by García-Cruz & Martinón (1996b).

In this paper we report briefly some results of our ongoing research focused on the students process of generalization when solving linear generalising problems. The research questions were:

a) Do students use a visual or a numerical strategy?

b) How do students check their patterns?

Generally, a *visual strategy* is defined as the method of solution that "involves visual imagery, with or without a diagram, as an essential part of the method of solution" (Presmeg, 1986, p.298). In this paper, a *visual strategy* is defined as one in which the drawing plays an essential role in the process of abstraction. A *numerical strategy* is defined, accordingly, as one in which the numerical sequence plays an essential role in the process of abstraction.

1 Theoretical Background

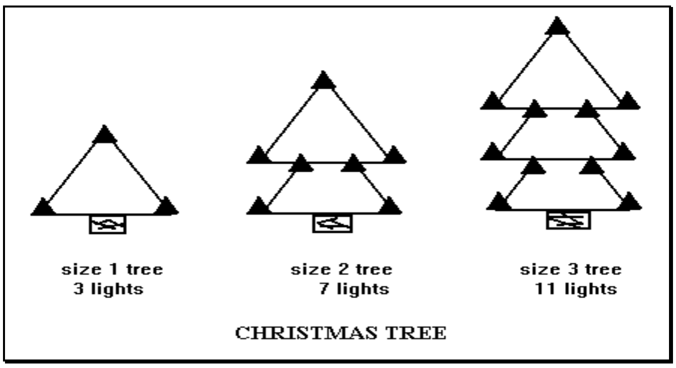
There is a broad agreement that the essential characteristic of mathematical knowledge is its generality and abstractness. Abstraction and generalization are important as a product but from a didactic point of view the associated processes of abstraction and generalization are much more important. W. Dörfler (1991) has modelled in detail the process of what Piaget called reflective abstraction, within this model the abstraction is the mean to construct a generalization. In our study we have adopted this theoretical model.

The essential features of this model are the emphasis on *actions* as the genetic source for abstraction and generalization. The *actions* that are material, imagined or symbolic are the starting point for the process of abstraction, even mathematical operations must be regarded as *actions*. Thus, the starting point is an *action* introduced by the student that concerns the elements given in the problem (either the drawing or the numerical sequence) as a response to questions which state an objective (calculate the numbers of components $f(n)$ for an object of a given size n). This *action* or *system of actions* directs the student's attention to some relations and connections between the elements of the *action*, size and components of the given object, and as a result to establish an *invariant* for the action.

This establishing of an *invariant* and its symbolic description has the character of a process of abstraction because some certain properties and relationships are pointed out and attention is focused upon them. Thus, the *action* or *system of actions* determines to some extent the directions and the content of the generalizations, i.e., the *invariants*, which operative character (the rules stated) results from the genesis out of the *actions*. To develop to a certain degree a generalization the student has to establish the *schema of the action* (*invariant*) as a general structure, i.e., to construct an *intensional generalization*. At this point the generality thereby constructed does not represent the qualities of things but relations between things, n and $f(n)$, which have been established and constructed by *actions*. The result of this process is a variable cognitive model that has two complementary aspects, first an expression of a cognitive activity of the subject and second as part of the objective knowledge, the mathematical content.

2 Methodology

The research was carried out in two stages. In the first stage a written test was administered to all students (N=168) in the last year of compulsory secondary education (aged 15-16 years) at a suburban high school and at the begining of the school year. For 133 out of 168 students the written test was the problem-1.



Problem-1

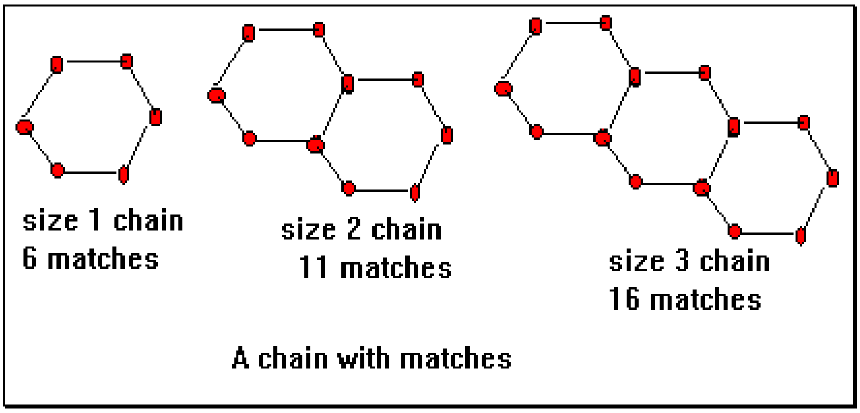
a)How many lights are there on a size 4 tree?
b)How many lights are there on a size 5 tree?
c)How many lights are there on a size 10 tree?
d) How many lights are there on a size 20 tree?

Explain how you found your answer.

In order to get a better analysis of the numerical strategies the following version of problem-1(with no drawing) was administered to a small group of students (N=35).

Problem -1a . *Ana and Juan are building up the Christmas Tree. In the instructions' booklet they found the following: A size 1 tree would need 3 lights, a size 2 tree would need 7 lights and a size 3 tree would need 11 lights. Try to help Ana and Juan answering the following questions:*
a) How many lights would need a size 4 tree?
b) How many lights would need a size 5 tree?
c) How many lights would need a size 10 tree?
d) How many lights would need a size 20 tree? Explain how you found your answer.

After the analysis of the written responses eight students were selected from the group were the problem-1 was administered (students *S1* to *S8*). Two of them shown in their responses that the drawing was used, two shown that only the numerical data were used and four gave no explanation or from their explanation not a clear conclusion could be obtained about the use of the drawing or the numerical data. From the second group, problem-1a, three students were chosen (student *S9*, *S10* and *S11*). The whole group of eleven students was chosen for the variety and quality of their responses to the written test. In the second stage these students were given individual interviews and asked first about some questions on problems 1 and 1a that may have not been clearly state from the written responses and second they were confronted with some questions about the situation stated in problem-2.



Problem-2

How many matches would you need to make the same sort of chain with size 4 ?

How many matches would you need to make the same sort of chain with size 23 ?

The objective of this second task was to verify *in situ* how students develop the process of abstraction and generalization and to what extent they recognize the second problem as similar to the first one. A consequence of this methodology was that the researchers did not ask necessarily the same questions to each student. Also they did not know if the students had received a specific instruction on arithmetic sequences, a topic related before, but they were aware that no student had had any training in sequences from the begining of the school year. Only two students belonging to the group interviewed had received specific instruction on arithmetic sequences the year before, but this fact was discovered during the interviews.

3 Results

The process of abstraction and generalization has actions introduced by students concerning the elements of the situations as its genetic source within the theoretical framework adopted in this study. The objective of these actions is to find out the number of elements *f(n)* corresponding to an object of size *n*.. Acting upon the numerical sequence or upon the drawing the elements of the actions are conceived as variables while certain relationship is maintained, i.e., the invariant. Now we will describe briefly the actions and the invariant schemata developed by these eleven students, so the list should not be considered exhaustive.

Actions

Concernig the drawing.

Drawing a picture of the whole object required and counting all the elements is an action used in the introductory questions, *f(4)* or *f(5),* and do not lead to a generalized strategy obviously. Students usually did it as a mean to check the validity of their calculations, as we will see below. Actions which lead to a generalized strategy are:

A1: Imaging or sketching to some extent a picture of the object required and adding similar parts while each new part has a number of elements equal to the constant difference *d*. The special feature here is that not direct counting at all is performed by students in the sketch done.

A2: Imaging an object of a certain size as constituted by aggregation of other objects of lesser size, i.e., a ten-size object as built up from two five-size objects.

Concerning the numerical data.

A3: Counting from a given term (i.e., *f(4)* but not *f(1)*) the number of *d* (the constant difference) which must be added to get a specific term (i.e., *f(10)*).

A4: Similar to action *A2* but performed upon the numerical sequence.

A5: Find a functional relationship between the object size *n* and the number of components *f(n)*.

A6: Applying the algorithm *rule-of-three*, which consists in giving three numbers to calculate a fourth number using the following schema:

5 —— 19
10 —— x

$$x = \frac{10\times 19}{5} = 38$$

Obviously the result of this calculation does not correspond with any term in the sequence but after doing that calculation the student *S3* checked it and made some arrangements leading her to get the correct answer. Below we will analyze in more detail the performance of this student. Here we have a system of actions instead of only one action.

A7: Applying the symbolic expresion for arithmetic sequences learned before. During the interview the student *S4* recognized the numerical pattern of problem-1 as arithmetic and after some calculations and checking he reconstructed the corresponding general symbolic expresion *f(n)=f(1)+(n-1)d*; later while he was confronted with problem-2 he applied automatically this formula showing an explicit knowledge of the similar structure of both problems.

A8: Successive addition of the constant difference to extend the numerical sequence.

Invariant schemata as result of actions

As result of the actions described above students established the following invariant schemata:

I1: f(n)=d(n-1)+f(1). Developed from actions upon the drawing, also from the numerical sequence.

I2: f(n)=6n-(n-1). This invariant was developed by *S2* within the problem-2. The important feature of this invariant is that neither the constant difference nor the first term in the sequence is an essential part of it.

Both invariants *I1* and *I2* were derived from action *A1*. Thus, the same action performed upon the drawing can lead to two different invariants. In *I1* the constant difference *d* and the first term *f(1)* play a prominent role while in *I2* both elements are not essential parts.

I3: f(n)=d(n-m)+f(m), m>1. This invariant was developed by student *S6* acting upon the drawing and by student *S9* acting upon the numerical sequence.

I4: f(2n)=2f(n).

I5: f(n)=dn+b. This invariant states the functional relationship between *n* and *f(n),* corresponding to *4n-1* and *5n+1* in problems 1 and 2 respectively.

I6: Derived from action *A6*. The student *S3* developed this invariant for problem-1 and here its symbolic expression corresponds with *f(2n)=2f(n)+1*.

I7: f(n)=dn. Derived from action *A8*, assuming that repeated addition of *d* implies *f(n)=dn*.

Table-I summarizes the correspondence between actions and invariant schemata established by the eleven students in our study:

Table-I	Problem-1 and 1a				Problem-2			
	Drawing		Number Sequence		Drawing		Number Sequence	
students	Act.	Inv.	Act.	Inv.	Act.	Inv.	Act.	Inv.
S1	*A1*	*I1*			*A1*	*I1*		
S2	*A1*	*I1*			*A1*	*I2*		
S3			*A6*	*I6*			*A6*	*I6*
S4			*A6*	*I1*			*A6*	*I1*
S5			*A4*	*I4*	*A1*	*I1*		
S6	*A1*	*I3*			*A2*	*I4*		
S7			*A5*	*I5*			*A5*	*I5*
S8	*A1*	*I1*					*several*	*no inv.*
S9	-----	-----	*A3*	*I3*			*A6*	*I1*
S10	-----	-----	*A4*	*I4*			*A8*	*I7*
S11	-----	-----	*A5*	*I5*			*A5*	*no inv.*

4 Discussion

Although students use more than one action we have placed in Table-I only the last action with which they have completed the process of establishing an invariant. To establish an invariant the student has to apply the same rule abstracted from a specific calculation, i.e., *f(4),* at least to another calculation, i.e., *f(10),* showing that he or she has made an *intensional generalization* (establishing the schema of the action as a general structure) and an *extensional generalization* (extending the range of *n*).

From Table-I we gather the following indications: Different invariants can be established from only one action, thus action *A1*performed upon the drawing leads to three different invariants, this is so because students' attention is focused in some aspects of the drawing highlighting these from other aspects. The actions performed upon the numerical sequence leads to the stating of only one invariant, due to the specific feature of the mathematical operation involved. Only two students out of eleven did not establish an invariant within the problem-2. A special case, student *S8*, will be discussed later.

Checking the rule abstracted should be considered an action as well. This action turns to be absolutely essential for students performing their calculation in the numerical setting as it is shown in Table-II.

Table II	where do students check	
	upon drawing	upon num. sequence
problem-1	*S3, S4, S6, S8*	*S5, S7*
problem-1a	------------	*S9, S11*
problem-2	*S1, S3, S5, S6, S9*	*S7, S8, S11*

If we compare Table-I and Table-II we can see that students performing their actions upon the drawing do not check their rules using the given numerical data. These students showed during the interviews more confidence that students performing their actions upon the numerical sequence. The actions upon the drawing fits the general structure of the rule in the students' cognition more precisely (intensional generalization), and the subsequent application of this rule to any other calculation was done with no doubt and confidently. However, some students whose actions were performed upon the numerical sequence (*S3* and *S4* within problem-1; *S3* and *S9* within problem-2) check the validity or their rules in the spatial setting, the drawing. Only two students, *S2* and *S10* did not check their rules during the process of solution in both problems. Student *S4* did not check his rule for problem-2 because he recognized the problem structure as similar to problem-1 and then he automatically applied the same invariant. He did not remember the symbolic expression for the general term of an arithmetic sequence, but he was able to reconstruct it while solving the problem. This case should be considered as an outstanding performance of developing a generalization.

The usual way of checking the validity of an invariant is counting on a drawing or extending the numerical sequence till the term needed. The use of routine activities for checking reinforces the students' confidence on the rule abstracted. Only two checks (student *S7* and *S11*) were done, using a known pair of values *(n, f(n))* and substituting this pair on the corresponding abstracted rule in both problems. This proved to be essential in establishing the invariant *I5* through the action *A5*.

An outstanding check was performed by student *S3*. She assimilates the calculation of *f(10)* in problem-1 to the existing schema, *rule-of-three*, using the pair *(5,19)* obtained adding the constant difference to *f(4)*. After doing the corresponding calculations she obtained the value 38 for *f(10)* and comparing this value to the available sequence 3,7,11,15,19 she notes that the number 38 (even number) does not fit in that sequence (odd numbers). Thus, she turns to draw a sketch of a size-ten-tree and performs a direct counting of the lights needed obtaining 39. Then she says that she must add one to the resulting calculations obtained applying the *rule-of-three*. For the next calculation, *f(20),* she applied the rule derived without checking on the drawing. She accommodated the particular schema of *rule-of-three* to fit the situation, establishing an invariant which symbolic expression is equivalent to $f(2n)=2f(n)+1$.

To develop a rule for a specific calculation it should no be considered as the establishing of an invariant. During the interview the performance of student *S8* should be considered as paradigmatic of the employ of many actions leading to the establishing of no invariant. He starts calculation of *f(4)* sketching a picture of a size-four-chain to end counting on the sketch to get $f(4)=f(3)+5=16+5=21$. When promted to calculate *f(15)* he applied the *rule-of-three* using data *(4, f(4))* but the outcome calculation was not a whole number, so after some numerical explorations he used action *A3* to get $f(15)=5\times11+f(4)=76$. Then he was promted to calculate *f(32)* and, instead of applying the rule developed before, he said that the solution should be $f(32)=2f(15)$ plus something else. At this point he was encouraged to reflect on previous calculations. Suddenly he starts action *A5* sketching roughly a table using the available data and after some trying he concluded the task with the expression $f(32)=32\times5+1$. It seems that he has established an invariant after all, but when asked about the validity of that rule for further calculations he said that he had not enough confidence in that rule unless he had tried it before, because that rule could fail in some case and then he would have to develop another one. The whole activity of student *S8* shows that neither intensional nor extensional generalization has been achieved. He has developed a specific rule for every calculation. The most perplexing thing for the interviewer was that he had successfully developed and established an invariant for problem-1, but this was done acting on the drawing instead on the numerical sequence. For this student the numerical setting probed to be harder than the spatial setting.

Otherwise, for student *S1* and *S2* the drawing probed to be the best setting for developing a generalization. Though an example, the calculation of *f(4),* they were able to establish an invariant. They felt very confidences with the rule abstracted and that rule was successfully applied to any further calculations. For these students, the variable elements in the rule abstracted are detached from their original values that they are associated and gain meaning by themselves.

The students' performance discussed shows that the role played by the drawing is twofold. First is the setting for developing (abstracting) a rule and second is the setting for checking the validity of a rule developed upon the numerical sequence.

5 Conclusions

The theoretical model (Dörfler, 1991) for generalization has proved useful for describing the students' developing of generalization in this type of problem. To establish an invariant could need many actions, or system of actions, for some students while other only need of one action.

We have distinguished within the process two key aspects: first to abstract a rule for a specific calculation and second to establish the general structure of this rule and to extend the range of the variable elements (intensional and extensional generalization). The same action could lead to different invariants and also the invariants could include (or not) essential elements (d and b) of the underlying mathematical object: the affin function $f(n)=dn+b$.

The students' use of learned knowledge, although not appropriate for these problems, is an important feature of students' performance. The case of the assimilation-accommodation of the *rule-of-three* leading to the establishment of an invariant is an outstanding student's behaviour. The consistency of students' choice of the numerical or spatial setting is a relevant conclusion derived from this study (see Table-I and Table-II).

Space limit does not allow us to extend in the analysis of the visual and numerical strategies, but we hope the above discussion can serve as an outline of our still not finished research in this area. It should be emphasized here the important role played be the drawing in the visual and numerical strategies, being the setting were students develop their rules during the process of abstraction in the former and being the setting for checking that rules in the later. We also think that more research is needed here to clarify in more detail the students' behaviour when establishing an invariant and the special features of the drawing that could lead a student to establish an invariant like *I2*.

Finally our ongoing research is now mainly directed to the study of the particular symbolizations students use in these type of problems, also to the meaning they give to the usual standard mathematical symbolization. But there is another research question: By what means do students recognize a similar mathematical structure among these type of problems?

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