Levels of generalization in linear patterns

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In this paper we set forth a proposal about levels of generalization based on students' spontaneous performance during the process of solving linear generalizations problems. The students' acquisition of each level is related to the actual generalization achieved and some features of the students' generalization process are provided. We conclude with a first approach to a genetic decomposition schema of the linear pattern's conceptual structure that students develop when solving linear generalizing problems. We also outline some didactic remarks that should be considered during the teaching and learning process.

Theoretical Background

The study of patterns' generalization in school mathematics has been the focus of research conducted over the last years. Many researchers have made some attempts to investigate stages or levels in the development of patterning ability mainly focused on students' ability to generalize. Stacey (1989) has identified some methods of solution that students use when solving linear generalizing problems. In Orton & Orton (1994, 1996) the adults' and children's answers to questions involving quadratic and linear patterns are classified in stages running from answering questions about concrete numbers to algebraic generalization. Redden (1994) has used the SOLO taxonomy to state two hierarchies of growth concerning first the students' use of data from the questions (Data processing dimension); and second the sense of an overview of the data that can be provided in the form of an expression of generality in the students' pattern description (Expressing Generality Dimension). However, we think that these attempts are mainly focused on the students' written responses to an item and to specific questions within an isolated item. Thus, the dynamic development of learning is not reflected enough. We have also missed a general framework to cope with the problem of students' pattern generalization that can be used not only to classify the students' responses but also to highlight some didactic guides to be used during the process of learning and teaching. Krutetskii (1976, p236) has pointed out that the ability to generalize mathematical material can be considered from two levels: *"(1) as a person's ability to see something general and known to him in what is particular and concrete* " (subsuming a particular case under a known general concept) and *"(2) the ability to see something general and still unknown to him in what is isolated and particular*" (to deduce the general from particular cases). The way the two levels are formulated shows that they do not constitute a hierarchy of students' educational development but both should be seen as educational goals. It is the second level, generalization through empirical induction, the ability we want to develop in students when they are dealing with new problematic situations.

The main goal of our research, we report in this paper, is to state some hierarchical levels of generalization that can reflect the students' performance when dealing with that kind of problems, and can also be used to provide some didactic remarks in helping students to move from one level to the next.

The role played by Reflexive Abstraction (Piaget, 1975) in the generalization's process has been the key feature of some recent research, for instance the action-process-object framework of Dubinsky(1991), and the operative generalization of Dörfler (1991). In our research we have taken the action-process-object framework from Dubinsky, in which the generalizations are constructed through the internal coordination of processes. These processes have their genetic sources in actions performed by the subject on a given stimulus, but we have added the key feature of Dörfler's theory, i.e., a generalization is achieved through the establishment of an invariant which genetic source is again an action performed by the subject. Briefly, a physical or mental action performed by the subject could lead to an internal process, and through coordination or reconstruction (assimilation-accommodation) of existing conceptual schemata, the subject could establish an invariant for the action. The generalization developed could take different forms depending on the actual kind of assimilation of the stimulus by the subject, and therefore different levels related to mathematical concept's achievement could be defined. On the other hand, what is actually achieved in any level could be used to derive some didactic remarks to be implemented during the process of teaching and learning.

Methodology

We have conducted our empirical research on a population of secondary education students (15-16 year olds). The first phase consisted in video-recorded interviews administered to eleven students. The second phase was an interactionist teaching experiment with a group of 18 students. Thus, from the interactionist perspective (Bauersfeld, 1994) there are neither pre-given criteria about what is a correct solution nor what constitutes a different solution to a give problematic situation. So students have to contribute to the whole-class discussion providing their own solutions to a problem, and give different solutions from the same problem. They were also encouraged to judge any solution presented to the whole class discussion. Our goal was that these sociomathematical norms (Yackel & Cobb, 1996) could help students to develop a better and deeper understanding of the linear pattern. We think that when a student assumes and uses explanation, judgement or argument as an object itself of discourse, he will require the development of methacognitive abilities that will improve the student's learning outcome.

During the four classroom sessions the students were presented with three linear generalizing problems (stimulus items). These problems underlie a linear pattern, $f(n)=an+b$, being $f(n)>0$, $a>0$, $b\neq 0$, whole numbers. The text format is a word problem illustrated by a drawing of an object and the first three terms of the sequence (number and drawing) are given, i.e., $f(1)$, $f(2)$ *and* $f(3)$, and students were asked to find $f(4)$, $f(5)$ (introductory questions) and $f(10)$, $f(20)$ and $f(n)$ later. Our role was to facilitate and encourage students' participation in small-group and whole-class discussion. When a solution was explained, we had to ask for any other students who wanted to judge it, being careful not to show disagreement or any kind of behaviour that could give any hint to the whole group about the correct or incorrect quality of that solution. Students were obliged to try to develop personally meaningful solutions that they could explain and justify, and reflection upon their own and others' strategies of solutions was encouraged.

Results and Discussion

Some early research (García-Cruz & Martinón, 1997a, 1997b) has provided us with useful information about the students' process of generalization. The actions developed and invariant schemata established during the process of solving a sequence of linear generalization problems are the key feature to achieve a generalization. Also, the conceptual schemata coordinated by the students are important to characterize each level. At each level, we have stated what previous schemata are coordinated and which generalization is achieved by students. Also these levels characterize the cognitive students' behaviour and can be used to distinguish between procedural activity, procedural understanding and conceptual understanding (Zazkis & Campbell, 1996). Our findings are summarized in a final developmental schema that can be seen as a genetic decomposition (Dubinsky & Lewin, 1986) of the linear pattern's cognitive structure through linear generalizing problems.

Level-1 (Procedural activity)

At this level, the student recognizes the iterative and recursive character of the linear pattern, and these are used to calculate the introductory questions. These strategies are not generalizable but are important in highlighting the constant difference of the linear pattern. Such a routine behaviour is later used (another level) when checking the validity of the rules developed. Here students are focused in the most perceptual feature of the pattern: *adding the constant difference* and this action is the only generalization achieved at this level. There is a subtle difference between the "counting all" strategy (f(10) = f(1)+d+...+d) and the "counting on" strategy (f(10)=f(9)+d): one thing is to add repeatedly the constant difference to get any term, extending the numerical sequence (iterative character) and another thing is to use the recursive character of the pattern using a known term and from this numerical value perform some calculations to get the required term. The term procedural activity could be used to characterize the student 's behaviour at this level.

Level-2 (Procedural understanding. Local Generalization)

At this level, the student has established a local generalization. This means that he or she has been able to establish an invariant from an action performed on the picture or numerical sequence, within any new problem given, although this invariant could be different from problem to problem. The establishing of the invariant means that the same calculation rule, derived from actions to calculate a specific term, has been applied to any other calculation within the same problem or situation (García-Cruz & Martinón, 1997a).

The establishing of an invariant also means that the stimulus has been assimilated and accommodated within an already existing cognitive schema, i.e. indirect counting methods, function (as a process), and proportional reasoning. The existing cognitive schema is identified by the student 's written of verbal response.

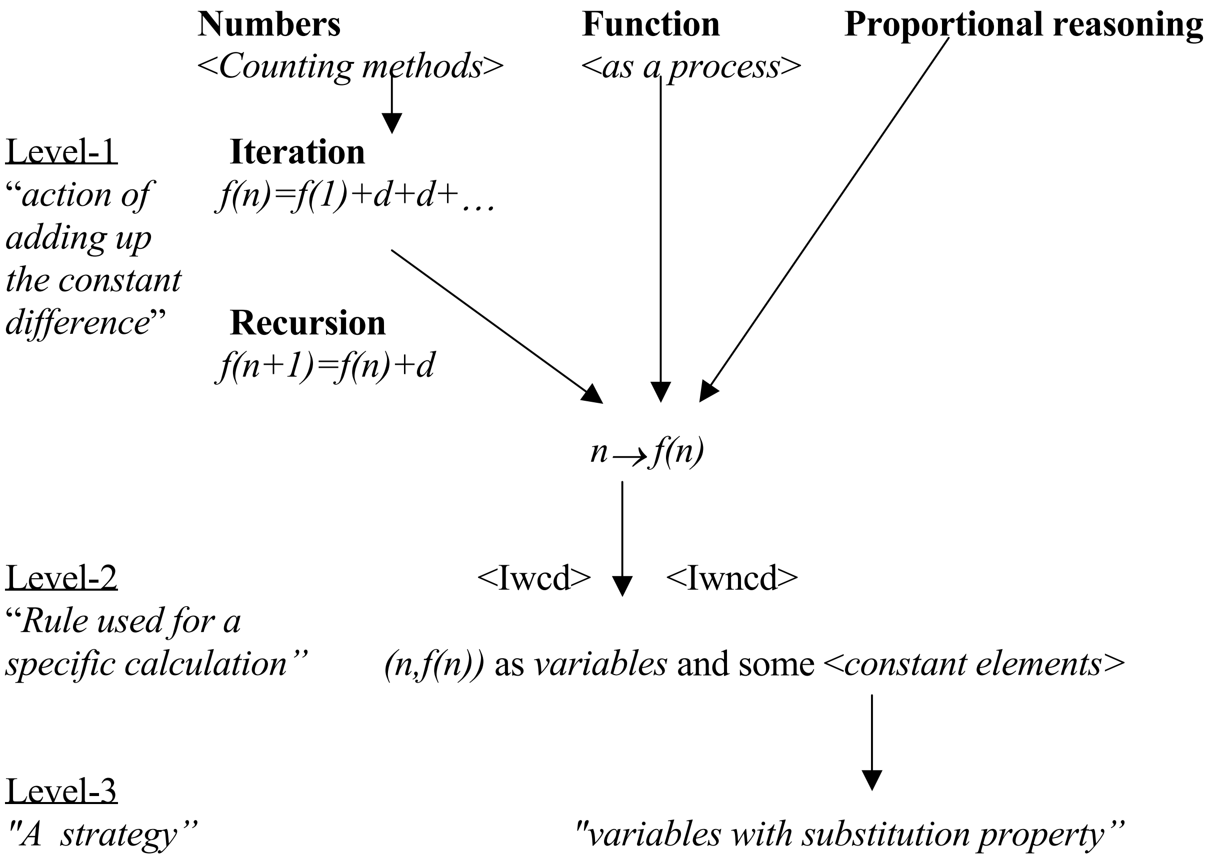
The student can also establish an incorrect invariant because the stimulus is assimilated to an incorrect cognitive schema, i.e., proportional reasoning. As we said above the establishing of an invariant is detected through the calculation rule used by the student in any question within a problem. If the canonical form of the linear pattern is $f(n)=5n-1$, then the assimilation of that stimulus to the incorrect cognitive schema of proportional reasoning could lead the student to the establishing of an invariant of the form $f(2n) = 2f(n)$. Later through checking and adjustment, this invariant could take the form $f(2n)=2f(n)-1$ which is valid only for even terms in the sequence. The student's attention can be also focused on some relations and connections between some elements of the drawing, and as a result an invariant of the form $f(n) = 6n-(n-1)$, or $f(n) = 6+5(n-1)$ can be established. So in establishing an invariant, students confer a variable quality to $f(n)$ and n, i.e., value of the term and position occupied in the corresponding sequence be numerical or pictorial.

The key feature here is that a shift from procedural activity to procedural understanding has taken place, and this shift can be clearly observed in the students performance. Thus, what has been generalized here is the *specific rule* for a calculation. This rule has always variable and non-variable elements, and the character confered to the variable elements should be taken as a generalization. Indeed, an extensional and intensional generalization has taken place, because the specific elements, numerical or pictorial, used to develop the rule have been detached from their initial meaning and their reference range has been extended. When the established invariant is correct the term procedural understanding could be used for the student's cognitive behaviour.

Level-3 (Conceptual understanding. Global Generalization)

At this level, the student has generalized a *strategy*. That means that he or she has performed the same action and established the same invariant in a new but similar problem. The *rule* developed and used in an early problem is now an object which serves as an stimulus for an action: apply or transfer the action performed and invariant established in another problem to a new problem which has been recognized as similar to other already known. At this level, what is achieved as a generalization is the student's overall performance when dealing with these situations, and this is what we call a *strategy*. So a strategy has the action and the invariant established as components in a particular situation. Now this strategy is used in a new but similar situation. The constant elements, if any, (which are present in the syntactic structure of the invariant) acquire the quality of variables through that process because they loose their constant character and are substituted by different numbers. The students cognitive behaviour could now be considered as conceptual understanding.

The following schema summarizes the above discussion and could be considered as a first approach to the genetic decomposition of the students' conceptual structure of linear patterns developed spontaneously by students though linear generalizing problems (Dubinsky & Lewin, 1986).



(Note: *Iwcd* and *Iwncd* stand for Invariant with constant difference and Invariant with no constant difference respectively)

The above schema should be taken as the ways in which students spontaneously develop their understanding of linear pattern's conceptual structure, but perhaps we think it is not complete.

From the teaching experiment we have drawn some conclusions.

First, it takes time before a student realizes that the existing conceptual structures are not sufficient to assimilate the new problematic situation, so many students keep on establishing incorrect invariant and it seems for us very difficoult to remove this unsuccessful behaviour. For those students we strongly recommende the generalization of conditions for actions (second process of Dörfler´s theory), so in order to carry out actions these are better if the constant difference are a key component.

Second, other students are successful in establishing an invariant (local generalization) but they move from one invariant to another (even incorrect) when confronted with a new situation. For those students we strongly recommend the generalization of the results of actions (third process of Dörfler's theory).

Third, once a local or global generalization has been achieved, the students should be confronted with a large number of new situations before the new cognitive structure becomes stable and permanent.

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